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Multi-period mean–variance portfolio optimization with management fees

Xiangyu Cui[1] · Jianjun Gao[2] · Yun Shi[3]

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Abstract
Due to limited capital and limited information from stock market, some individual investors prefer to construct a portfolio of funds instead of stocks. But, there will be management fees paid to the fund managers during the investment, which are in general proportional to the net asset value of the funds. Motivated by this phenomena, this paper considers multi-period mean–variance portfolio optimization problem with proportional management fees. Using stochastic dynamic programming, we derive the semi-analytical optimal portfolio policy. Our result helps clarify the benefit and cost of adopting such dynamic portfolio policy with management fees.

Keywords Dynamic mean–variance portfolio selection · Management fee · Dynamic programming

1 Introduction

Markowitz (1952)'s seminar work on mean–variance (MV) portfolio selection theory laid the foundation of modern investment theory. This classical model has been widely adopted in both theoretical study and financial practice. As two of the most important extensions of the traditional (static) MV model, Li and Ng (2000), Zhou and Li (2000) derived the analytical dynamic portfolio policies for the discrete-time and continuous-time models, respectively. Following these works, the multi-period MV portfolio policies under different constraints and market parameter settings are derived, e.g., Zhu et al. (2004), Çakmak and Özekici (2006), Costa and Araujo

✉ Yun Shi
yshi@fem.ecnu.edu.cn

[1] School of Statistics and Management, Shanghai Institute of International Finance and Economics, Shanghai University of Finance and Economics, Shanghai, China

[2] School of Information Management and Engineering, Shanghai University of Finance and Economics, Shanghai, China

[3] Institute for Statistics and Interdisciplinary Sciences, Faculty of Economics and Management, East China Normal University, Shanghai, China

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(2008), Cvitanić et al. (2008), Chiu and Zhou (2011), Cui et al. (2012), Cui et al. (2014), Cui et al. (2017), Keykhaei (218). Besides these multi-period MV portfolio policies, there are also dynamic portfolio policies involving other features, such as views at multiple horizons (Meucci and Nicolosi 2016), linear rebalancing rules (Moallemi and Sağlam 2015), multiple criteria (Pendaraki and Zopounidis 2003, Xidonas et al. 2009, Yu and Lee 2011), investor attitudes and biases (Momen et al. 2017).

In portfolio management practice, some individual investors prefer to construct the portfolios with different funds instead of stocks, due to limited capital and limited information from stock market. If the investor believes that the professional fund managers can do better by taking advantage of the predicting power in the stock market, he would like to construct his portfolio of different funds. However, when investing in fund, there will be management fees paid to the fund managers. In general, the management fees are proportional to the net asset value of the funds. Comparing to the rich literature in dynamic portfolio selection, there are few papers considering management fees. Brown et al. (2004) studied the importance of management fees, Dokuchaev (2010) analyzed the myopic strategies, Gao et al. (2015) considered only the setup type of management fee.

In this work, we focus on the multi-period MV portfolio selection problem with the common proportional type of management fees. The problem can be reformulated as a particular multi-period MV portfolio selection problem with no-shorting constraint. By using the technique proposed in Cui et al. (2014), we derived the semi-analytical optimal portfolio policy for this problem under general model of fund returns. We show that the revealed portfolio policy is a linear threshold-type policy, which is significantly different from the traditional linear MV portfolio policy. Our results may also help the investor to understand the benefit and cost of adopting dynamic portfolio policy with management fees.

A related subject in the literature is the optimal investment and consumption problem with transaction costs. Transaction costs are charged when there is a holding position change in risky stocks, while the management fee is charged when the fund is being held. In the language of math, if transaction costs are involved in a dynamic portfolio selection problem, the investor needs to keep track of portfolio positions across periods. But, if management fees are involved, only the current wealth level is needed to be known. In the literature, there are many studies on portfolio selection problem with transaction costs in continuous-time and discrete-time settings, e.g., Dai et al. (2010), Liu (2004), Bertsimas and Pachamanova (2008), Lynch and Tan (2010), Yu and Lee (2011), Garleanu and Pedersen (2013). Another less related field is the delegated portfolio management problem, which focuses on the incentives and risk sharing between principal and agent (e.g., Laffont and Martimort 2002; Ou-Yang 2003; Li and Tiwari 2009; Sato 2016). Readers may refer to Stracca (2006) for a review for this field.

The remaining of this paper is organized as follows. In Sect. 2, the multi-period mean–variance formulation with proportional management fees is presented. In Sect. 3, we derive the semi-analytical portfolio policy for the problem. In Sect. 4, we provide an illustrative example to study the properties of the optimal policy. We conclude the paper in Sect. 5.

2 Problem formulation

Through out the paper, we use the following notations. We use $A \succ 0$ to denote a positive definite matrix A and $\mathbf{v}'$ to denote the transpose of vector $\mathbf{v}$. Let $\mathbf{1}_{\mathcal{B}}$ be the indicator function such that $\mathbf{1}_{\mathcal{B}} = 1$ if event $\mathcal{B}$ held and $\mathbf{1}_{\mathcal{B}} = 0$, otherwise.

There are n different risky funds and one riskless bank account in the market, all of which evolve within a time horizon of T periods. An individual investor with an initial wealth x_0 joins the market at time 0 and allocates his wealth among the n risky funds and one riskless bank account at the beginning of each of the following T consecutive time periods. We denote the total return of the riskless bank account as a deterministic number s_t, $t = 0, \ldots, T-1$ (i.e., the ratio of the value of the bank account at time $t+1$ to the value of the bank account at time t), for simplicity, although there is no technical difficulty to extend it to the random case. The random total return vector of the n risky funds in the t-th time period are denoted as $\mathbf{e}_t = (e_t^1, \ldots, e_t^n)'$ (i.e., the ratio of the value of the risky funds at time $t+1$ to the value of the risky funds at time t), which is square integrable. In our study, we assume total return vectors in different time periods, $\{\mathbf{e}_t\}|_{t=0}^{T-1}$, to be statistically independent.[1] All the random vectors are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The information set at the beginning of the t-th time period is denoted as $\mathcal{F}_t = \sigma(\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{t-1})$ and $\mathcal{F}_0$ is the trivial σ-algebra over Ω. We use the notations $\mathrm{E}_t[\cdot]$, $\mathrm{Cov}_t[\cdot]$ and $\mathrm{Var}_t[\cdot]$ to denote the conditional expectation $\mathrm{E}[\cdot|\mathcal{F}_t]$, the conditional covariance matrix $\mathrm{Cov}[\cdot|\mathcal{F}_t]$ and the conditional variance $\mathrm{Var}[\cdot|\mathcal{F}_t]$, respectively. At time $t = 0$, the unconditional expectation and variance are denoted as $\mathrm{E}[\cdot] = \mathrm{E}[\cdot|\mathcal{F}_0]$ and $\mathrm{Var}[\cdot] = \mathrm{Var}[\cdot|\mathcal{F}_0]$, respectively. We assume that the covariance matrices $\mathrm{Cov}[\mathbf{e}_t]$, $t = 0, \ldots, T-1$, are all positive definite.

Let x_t be the investor's wealth level at the beginning of the t-th time period, $u_t^i \geq 0$ $(i = 1, 2, \ldots, n)$, be the amount of money in the long position of the i-th risky fund at the beginning of the t-th time period, and $v_t^i \geq 0$ $(i = 1, 2, \ldots, n)$, be the amount of money in the short position of the i-th risky fund at the beginning of the t-th time period. In practice, some brokers do allow their fund investors to sell the funds short, such as Jack White & Company, a discount brokerage based in San Diego, and Fidelity Investments in Boston (see Gould 1992). But there is much stricter margin requirement than selling stocks short. Thus, we allow the investor to sell the funds short by introducing the variables v_t^i in our setting. The proportional management fee of the i-th risky fund in the t-th time period is charged according to $c_t^i u_t^i + d_t^i v_t^i$, where c_t^i and d_t^i are two known positive constants representing the management fee rate. In general, the management fee of shorting a fund is much larger than the one of longing a fund, i.e., $d_t^i > c_t^i$.

Compared with taking a net position $u_t^i - v_t^i$ in fund i, taking both long position $u_t^i > 0$ and short position $v_t^i > 0$ in fund i can only generate an additional loss due to larger management fee. Thus, for most readers, it is reasonable to assume that the investor only takes either long position or short position in a fund, i.e., $u_t^i v_t^i = 0$. However, in the multi-period mean–variance framework, such irrational investment

[1] The main results in this paper can be readily extended to the case of correlated return vectors by adopting the technique proposed in Gao et al. (2015).

behavior, which causes wealth reduction, may generate better mean and variance pair of the terminal wealth. To reveal such interesting irrational phenomena, we assume that the investor is allowed to take both long position and short position in the same risky fund simultaneously in our model setting (i.e., both u_t^i and v_t^i take positive numbers at the same time).[2] In Proposition 2, we show that when the wealth level is smaller than some threshold, the investor only takes either long position or short position. When the wealth level is larger than the threshold, the investor takes equal long position and short position in order to reduce the wealth and finally decrease the variance of the terminal wealth.

Let $\mathbf{u}_t = \left(u_t^1, \ldots, u_t^n\right)'$ and $\boldsymbol{v}_t = \left(v_t^1, \ldots, v_t^n\right)'$. An investor is seeking the best $\mathcal{F}_t$-adapted portfolio policy, $\{\mathbf{u}_t^*, \boldsymbol{v}_t^*\}|_{t=0}^{T-1}$, such that the variance of the terminal wealth, $\mathrm{Var}[x_T]$, is minimized while the expected terminal wealth, $\mathrm{E}[x_T]$, is guaranteed at a given level b, with $b \geq x_0 \prod_{t=0}^{T-1} s_t$,

$$
\begin{aligned}
(\mathcal{P}(b)):\ \min\ & \mathrm{Var}[x_T] \triangleq \mathrm{E}\left[(x_T - b)^2\right],\\
\text{s.t. } & \mathrm{E}[x_T] = b,\\
& x_{t+1} = s_t(x_t - \mathbf{c}_t'\mathbf{u}_t - \mathbf{d}_t'\boldsymbol{v}_t) + \mathbf{P}_t'(\mathbf{u}_t - \boldsymbol{v}_t),\\
& \mathbf{u}_t \geq \mathbf{0}_n, \boldsymbol{v}_t \geq \mathbf{0}_n, \quad t = 0, 1, \ldots, T-1,
\end{aligned} \tag{1}
$$

where $\mathbf{P}_t = \left(P_t^1, P_t^2, \ldots, P_t^n\right)' = \left((e_t^1 - s_t), (e_t^2 - s_t), \ldots, (e_t^n - s_t)\right)'$ is the vector of excess rates of return, $\mathbf{c}_t = (c_t^1, c_t^2, \ldots, c_t^n)'$ and $\mathbf{d}_t = (d_t^1, d_t^2, \ldots, d_t^n)'$ are the vectors of management fee rates, and $\mathbf{0}_n$ denotes the n-dimensional zero vector. Except for the management fees, problem $(\mathcal{P}(b))$ is just the multi-period mean–variance portfolio selection problem. Readers may refer to Zhang et al. (2018) for review of mean–variance framework. Please note that the management fees are calculated and deducted at the beginning of each period. Furthermore, we assume that the market does not have arbitrage opportunity.

Once the wealth level at time t is negative, i.e., $x_t < 0$, the investor goes bankrupt. Thus, the constraint $x_t \geq 0$ (or $P(x_t < 0) \leq \alpha$) is often added into the portfolio selection model, which is called no bankruptcy constraint (see Bielecki et al. 2005; Zhu et al. 2004). In continuous-time setting, the investment policy can change quickly during a very short time interval and the constraint $x_t \geq 0$ can be easily fulfilled. Thus, Bielecki et al. (2005) directly handled the constraint $x_t \geq 0$. However, in multi-period setting, the investment policy can only change at several time instants and adding the constraint $x_t \geq 0$ may cause the feasible set empty. For example, we assume $\mathbf{P}_t$ has a normal distribution and $b > \prod_{t=0}^{T-1} s_t x_0$. Then, to ensure $x_{t+1} \geq 0$ for all possible realisations of $\mathbf{P}_t$, we need $(\mathbf{u}_t - \boldsymbol{v}_t) = \mathbf{0}$, which causes $\mathrm{E}\,[x_T] = \prod_{t=0}^{T-1} s_t x_0 < b$. There is no feasible policy for problem $(\mathcal{P}(b))$. Based on

[2] Based on the proofs of Proposition 1 and Theorem 2, the key requirement of applying our technique is that the admissible set of the control variables is a cone. $\{(u_t^i, v_t^i) | u_t^i \geq 0, v_t^i \geq 0, u_t^i v_t^i = 0\}$ is still a cone. Thus, our technique is also applicable to the setting that the investor can only take either long position or short position on a fund.

this finding, Zhu et al. (2004) used the constraint $P(x_t < 0) \leq \alpha$ to replace the constraint $x_t \geq 0$. In this paper, we do not consider the no bankruptcy constraint.[3]

If we define

$$\hat{\mathbf{P}}_t \triangleq \begin{pmatrix} \mathbf{P}_t - s_t \mathbf{c}_t \\ -\mathbf{P}_t - s_t \mathbf{d}_t \end{pmatrix}, \ \hat{\mathbf{u}}_t \triangleq \begin{pmatrix} \mathbf{u}_t \\ \boldsymbol{v}_t \end{pmatrix},$$

problem $(\mathcal{P}(b))$ can be reformulated as follows,

$$\begin{aligned} \min \ & E\left[(x_T - b)^2\right], \\ \text{s.t. } & \mathrm{E}[x_T] = b, \\ & x_{t+1} = s_t x_t + \hat{\mathbf{P}}_t' \hat{\mathbf{u}}_t, \\ & \hat{\mathbf{u}}_t \geq \mathbf{0}_{2n}, \quad t = 0, 1, \ldots, T-1. \end{aligned}$$

It is a particular multi-period mean–variance portfolio selection problem under no-shorting constraint. The technique proposed in Cui et al. (2014) can be applied to this reformulation. However, we still need to investigate the problem carefully due to the particular structure of $\hat{\mathbf{P}}_t$.

Assumption 1 (*Non-empty feasible set assumption*) There exists an $\mathcal{F}_t$-adapted portfolio policy such that the conditions (1) in problem $(\mathcal{P}(b))$ are satisfied. In other words, the feasible set of problem $(\mathcal{P}(b))$ is non-empty.

Remark 1 We can find two sufficient market conditions under which the feasible set of problem $(\mathcal{P}(b))$ is non-empty. The first one is that $\mathrm{E}[P_t^i] > s_t c_t^i$ holds for some risky fund i during certain period t. Then, the portfolio policy satisfying

$$u_t^i = \left(b - x_0 \prod_{t=0}^{T-1} s_t \right) \prod_{\ell=t+1}^{T-1} s_\ell^{-1} (\mathrm{E}[P_t^i] - s_t c_t^i)^{-1} > 0,$$

$v_t^i = 0, u_t^j = v_t^j = 0 \ (j \neq i)$ and $\mathbf{u}_s = \boldsymbol{v}_s = \mathbf{0}_n \ (s \neq t)$ is a feasible portfolio policy. The second one is that $\mathrm{E}[-P_t^i] > s_t d_t^i$ holds for some risky fund i during certain period t. Then, the portfolio policy satisfying

$$v_t^i = \left(b - x_0 \prod_{t=0}^{T-1} s_t \right) \prod_{\ell=t+1}^{T-1} s_\ell^{-1} (\mathrm{E}[-P_t^i] - s_t d_t^i)^{-1} > 0,$$

$u_t^i = 0, u_t^j = v_t^j = 0 \ (j \neq i)$ and $\mathbf{u}_s = \boldsymbol{v}_s = \mathbf{0}_n \ (s \neq t)$ is a feasible portfolio policy. The financial meaning of these two sufficient conditions is that after deducting the management fees, taking the long position (or the short position) in risky fund i may still earn positive expected excess return.

[3] As the no bankruptcy constraint is a state constraint, our technique is not applicable to such constraint.

Before presenting the main result, we need the following facts. Since $\mathrm{Cov}\left[\mathbf{e}_t\right] \succ 0$, we have $\mathrm{E}[\mathbf{e}_t\mathbf{e}_t'] = \mathrm{Cov}\left[\mathbf{e}_t\right] + \mathrm{E}[\mathbf{e}_t]\mathrm{E}[\mathbf{e}_t'] \succ 0$. Then, by applying Sherman–Morrison Formula, we further have

$$\mathrm{E}[\mathbf{e}_t']\big(\mathrm{E}[\mathbf{e}_t\mathbf{e}_t'] - \mathrm{E}[\mathbf{e}_t]\mathrm{E}[\mathbf{e}_t']\big)^{-1}\mathrm{E}[\mathbf{e}_t] = \frac{\mathrm{E}[\mathbf{e}_t']\mathrm{E}^{-1}[\mathbf{e}_t\mathbf{e}_t']\mathrm{E}[\mathbf{e}_t]}{1 - \mathrm{E}[\mathbf{e}_t']\mathrm{E}^{-1}[\mathbf{e}_t\mathbf{e}_t']\mathrm{E}[\mathbf{e}_t]} > 0,$$

which implies $1 - \mathrm{E}[\mathbf{e}_t']\mathrm{E}^{-1}[\mathbf{e}_t\mathbf{e}_t']\mathrm{E}[\mathbf{e}_t] > 0$. Combining $\mathrm{E}[\mathbf{e}_t\mathbf{e}_t'] \succ 0$ and

$$s_t^2 - s_t^2\mathrm{E}[\mathbf{e}_t']\mathrm{E}^{-1}[\mathbf{e}_t\mathbf{e}_t']\mathrm{E}[\mathbf{e}_t] > 0,$$

we can show that

$$\begin{pmatrix} s_t^2 & s_t\mathrm{E}[\mathbf{e}_t'] \\ s_t\mathrm{E}[\mathbf{e}_t] & \mathrm{E}[\mathbf{e}_t\mathbf{e}_t'] \end{pmatrix} \succ 0,$$

holds with the help of Schur complement condition for positive definiteness. Then, we have

$$\begin{pmatrix} s_t^2 & s_t\mathrm{E}[\mathbf{P}_t'] \\ s_t\mathrm{E}[\mathbf{P}_t] & \mathrm{E}[\mathbf{P}_t\mathbf{P}_t'] \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}_n' \\ -\mathbf{1}_n & I_n \end{pmatrix}\begin{pmatrix} s_t^2 & s_t\mathrm{E}[\mathbf{e}_t'] \\ s_t\mathrm{E}[\mathbf{e}_t] & \mathrm{E}[\mathbf{e}_t\mathbf{e}_t'] \end{pmatrix}\begin{pmatrix} 1 & -\mathbf{1}_n' \\ \mathbf{0}_n & I_n \end{pmatrix} \succ 0,$$

where $-\mathbf{1}_n$ is n-dimensional vector of all negative one and I_n is n-dimensional identity matrix. Applying Schur complement condition for positive definiteness again, we have

$$\begin{aligned} &\mathrm{E}[\mathbf{P}_t\mathbf{P}_t'] \succ 0, \quad t = 0, 1, \dots, T-1, \\ &s_t^2(1 - \mathrm{E}[\mathbf{P}_t']\mathrm{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathrm{E}[\mathbf{P}_t]) > 0, \quad t = 0, 1, \dots, T-1. \end{aligned}$$

3 Optimal portfolio policy with management fees

In the literature, the multi-period mean–variance portfolio policy is derived by the embedding method introduced by Li and Ng (2000), which is also equivalent to the Lagrangian duality method. In the following part, we generalize such method for problem $(\mathcal{P}(b))$ by introducing Lagrangian multiplier 2μ for constraint (1),

$$\begin{aligned} (\mathcal{L}(\mu)):\ \ & \min\ \mathrm{E}\left[(x_T - b)^2 + 2\mu(x_T - b)\right], \\ & \text{s.t.}\ \ \{x_t, \mathbf{u}_t, \boldsymbol{v}_t\}\ \text{satisfy (2), (3).} \end{aligned}$$

The following Theorem shows that the strong duality relationship holds for $(\mathcal{P}(b))$ and $(\mathcal{L}(\mu))$.

Theorem 1 *The strong duality relationship holds for problems $(\mathcal{P}(b))$ and $(\mathcal{L}(\mu))$, i.e.,*

$$v(\mathcal{P}(b)) = \max_{\mu\in\mathbb{R}} v(\mathcal{L}(\mu)), \tag{2}$$

where $v(\mathcal{P}(b))$ and $v(\mathcal{L}(\mu))$ denote the optimal objective values of problems $(\mathcal{P}(b))$ and $(\mathcal{L}(\mu))$, respectively.

Proof Let $\pi^p = \{\mathbf{u}_t^p, \mathbf{v}_t^p\}|_{t=0}^{T-1}$ and $\pi^l = \{\mathbf{u}_t^l, \mathbf{v}_t^l\}|_{t=0}^{T-1}$ be the optimal policies of problems $(\mathcal{P}(b))$ and $(\mathcal{L}(\mu))$, respectively. The weak duality relationship always holds, i.e., for any μ, we have $v(\mathcal{L}(\mu)) \le v(\mathcal{P}(b))$, since π^p is always a feasible policy of problem $(\mathcal{L}(\mu))$ and $\mathrm{E}[x_T - b] = 0$ under policy π^p.

We then focus on the other direction. For any portfolio policy, $\pi \triangleq \{\mathbf{u}_t, \mathbf{v}_t\}|_{t=0}^{T-1}$, due to the dynamic in (1), we have

$$x_T = \prod_{t=0}^{T-1} s_t x_0 + \sum_{t=0}^{T-1} \prod_{\tau=t+1}^{T-1} s_\tau \left[\mathbf{P}_t'(\mathbf{u}_t - \mathbf{v}_t) - s_t \mathbf{c}_t' \mathbf{u}_t - s_t \mathbf{d}_t' \mathbf{v}_t\right]. \tag{3}$$

We then define the following functionals with respect to π, $\mathcal{F}(\pi) = \mathrm{E}[x_T] - b$ and $\mathcal{G}(\pi) = \mathrm{Var}[x_T] = \mathrm{E}[(x_T - b)^2]$. We can define the following set

$$\mathcal{D} \triangleq \left\{(\eta, \rho) \in \mathbb{R}^2 \mid \exists\, \pi, \text{satisfies } \mathcal{F}(\pi) = \eta,\ \mathcal{G}(\pi) \le \rho \text{ and conditions (2), (3)}\right\}.$$

We first show that set $\mathcal{D}$ is convex and not empty in $\mathbb{R}^2$. Suppose that $(\hat{\eta}, \hat{\rho}) \in \mathcal{D}$ and $(\bar{\eta}, \bar{\rho}) \in \mathcal{D}$, i.e., there exists $\hat{\pi} = \{\hat{\mathbf{u}}_t, \hat{\mathbf{v}}_t\}|_{t=0}^{T-1}$ and $\bar{\pi} = \{\bar{\mathbf{u}}_t, \bar{\mathbf{v}}_t\}|_{t=0}^{T-1}$ such that $\mathcal{F}(\hat{\pi}) = \hat{\eta}$, $\mathcal{G}(\hat{\pi}) \le \hat{\rho}$ and $\mathcal{F}(\bar{\pi}) = \bar{\eta}$, $\mathcal{G}(\bar{\pi}) \le \bar{\rho}$. Let $\check{\eta} = \alpha\hat{\eta} + (1-\alpha)\bar{\eta}$ and $\check{\rho} = \alpha\hat{\rho} + (1-\alpha)\bar{\rho}$ for any $\alpha \in [0, 1]$. We now prove that $(\check{\eta}, \check{\rho}) \in \mathcal{D}$. Let $\check{\pi} = \alpha\hat{\pi} + (1-\alpha)\bar{\pi}$. Then, from (3), it has

$$\begin{aligned}\mathcal{F}(\check{\pi}) = &\prod_{t=0}^{T-1} s_t x_0 - b + \sum_{t=0}^{T-1} \prod_{\tau=t+1}^{T-1} s_\tau \mathrm{E}\Big[\mathbf{P}_t'(\alpha\hat{\mathbf{u}}_t + (1-\alpha)\bar{\mathbf{u}}_t - \alpha\hat{\mathbf{v}}_t - (1-\alpha)\bar{\mathbf{v}}_t) \\ &- s_t \mathbf{c}_t'(\alpha\hat{\mathbf{u}}_t + (1-\alpha)\bar{\mathbf{u}}_t) - s_t \mathbf{d}_t'(\alpha\hat{\mathbf{v}}_t + (1-\alpha)\bar{\mathbf{v}}_t)\Big] \\ = &\ \alpha\hat{\eta} + (1-\alpha)\bar{\eta} = \check{\eta}.\end{aligned}$$

Let $\hat{x}_T$, $\bar{x}_T$ and $\check{x}_T$ be the resulting terminal wealth levels of policies $\hat{\pi}$, $\bar{\pi}$ and $\check{\pi}$, respectively. By using (1) and (3), we have

$$\begin{aligned}\mathcal{G}(\check{\pi}) = \mathrm{E}[(\alpha\hat{x}_T + (1-\alpha)\bar{x}_T - b)^2] &= \mathrm{E}\left[\left(\alpha(\hat{x}_T - b) + (1-\alpha)(\bar{x}_T - b)\right)^2\right] \\ &\le \alpha\mathrm{E}[(\hat{x}_T - b)^2] + (1-\alpha)\mathrm{E}[(\bar{x} - b)^2] \\ &= \alpha\mathcal{G}(\hat{\pi}) + (1-\alpha)\mathcal{G}(\bar{\pi}) \le \check{\rho}.\end{aligned}$$

Thus, we have found the policy $\check{\pi}$ such that $\mathcal{F}(\check{\pi}) = \check{\eta}$, $\mathcal{G}(\check{\pi}) \le \check{\rho}$, which further implies $(\check{\eta}, \check{\rho}) \in \mathcal{D}$ and $\mathcal{D}$ is convex. The non-empty property of $\mathcal{D}$ is from the Non-empty feasible set assumption.

Now, we consider another set $\mathcal{O} \triangleq \{(0, s) \in \mathbb{R}^2 \mid s < v(\mathcal{P}(b))\}$, which is a convex set. Note that $\mathcal{D} \bigcap \mathcal{O} = \emptyset$. We prove it by contradiction. Suppose $(\eta, \rho) \in \mathcal{D} \bigcap \mathcal{O}$. Since $(\eta, \rho) \in \mathcal{O}$, we have $\eta = 0$ and $\rho < v(\mathcal{P}(b))$. On the other hand, $(0, \rho) \in \mathcal{D}$. There exists π^* such that $\mathcal{F}(\pi^*) = 0$, $\mathcal{G}(\pi^*) \le \rho < v(\mathcal{P}(b))$, which contradicts the optimality of $v(\mathcal{P}(b))$. By using the separating hyperplane theorem of two convex sets [see

Theorem 11.3 on Page 93 in Rockafellar (1970)], we can find $(\bar{\mu}, \xi) \neq (0,0)$ and ζ such that

$$\bar{\mu} z + \xi y \geq \zeta \text{ for all } (z,y) \in \mathcal{D} \text{ and } \bar{\mu} z + \xi y \leq \zeta \text{ for all } (z,y) \in \mathcal{O}. \tag{4}$$

We can observe that $\xi \geq 0$, otherwise $\bar{\mu} z + \xi y$ can be $-\infty$ for $(z,y) \in \mathcal{D}$, which contradicts (4). For the case $\xi \neq 0$, we can define $\mu = \bar{\mu}/\xi$, which gives $\mu z + y \geq \zeta/\xi$ for all $(z,y) \in \mathcal{D}$ and $\mu z + y \leq \zeta/\xi$ for all $(z,y) \in \mathcal{O}$. Thus, from the definition of $\mathcal{D}$ and these two inequalities, for any π, we have $\mathrm{E}[(x_T - b)^2] + \mu(\mathrm{E}[x_T] - b) \geq \zeta/\xi \geq v(\mathcal{P}(b))$. Maximizing the left hand side of this inequality with respect to μ, gives $\max_\mu v(\mathcal{L}(\mu)) \geq v(\mathcal{P}(b))$. Combining with the weak duality $v(\mathcal{L}(\mu)) \leq v(\mathcal{P}(b))$, we have $v(\mathcal{L}(\mu)) = v(\mathcal{P}(b))$. When $\xi = 0$, it is not hard to see this relationship still holds. □

From Theorem 1, we can characterize the portfolio policy of problem $(\mathcal{P}(b))$ by solving problem $(\mathcal{L}(\mu))$. Note that problem $(\mathcal{L}(\mu))$ is equivalent to the following formulation by variable substitution,

$$\begin{aligned} (\mathcal{A}(\mu)) :\ & \min\ \mathrm{E}\left[y_T^2\right], \\ & \text{s.t. } y_{t+1} = s_t y_t + \hat{\mathbf{P}}_t' \hat{\mathbf{u}}_t, \\ & \qquad \hat{\mathbf{u}}_t \geq \mathbf{0}_{2n}, \quad t = 0, 1, \ldots, T-1, \\ & \qquad y_0 = x_0 - \rho_0^{-1}(b - \mu), \end{aligned}$$

where $\mathbf{0}_{2n}$ denotes the $2n$-dimensional zero vector, $y_t \triangleq x_t - \rho_t^{-1}(b - \mu)$, $\rho_t \triangleq \prod_{\ell=t}^{T-1} s_\ell$ (with $\prod_{\ell=T}^{T-1} s_\ell$ setting to 1), and

$$\hat{\mathbf{P}}_t \triangleq \begin{pmatrix} \mathbf{P}_t - s_t \mathbf{c}_t \\ -\mathbf{P}_t - s_t \mathbf{d}_t \end{pmatrix}, \ \hat{\mathbf{u}}_t \triangleq \begin{pmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{pmatrix}.$$

We use $v(\mathcal{A}(\mu))$ to denote the optimal objective value of problem $(\mathcal{A}(\mu))$. It is easy to check that $v(\mathcal{A}(\mu)) = v(\mathcal{L}(\mu)) + \mu^2$.

Proposition 1 *The optimal portfolio policy of* $(\mathcal{A}(\mu))$ *at time t is a piecewise linear policy given by*

$$\hat{\mathbf{u}}_t^* = s_t \mathbf{K}_t^+ y_t \mathbf{1}_{\{y_t \geq 0\}} - s_t \mathbf{K}_t^- y_t \mathbf{1}_{\{y_t < 0\}},$$

where

$$\mathbf{K}_t^- = \arg\min_{\mathbf{K}_t \geq \mathbf{0}_{2n}} \mathrm{E}\left[C_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}} + D_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \geq 1\}}\right], \tag{5}$$

$$\begin{aligned} \mathbf{K}_t^+ &= \arg\min_{\mathbf{K}_t \geq \mathbf{0}_{2n}} \mathrm{E}\left[C_{t+1}(1 + \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < -1\}} + D_{t+1}(1 + \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \geq -1\}}\right], \\ C_t &= \mathrm{E}\left[C_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t^-)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^- < 1\}} + D_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t^-)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^- \geq 1\}}\right], \\ D_t &= \mathrm{E}\left[C_{t+1}(1 + \hat{\mathbf{P}}_t'\mathbf{K}_t^+)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^+ < -1\}} + D_{t+1}(1 + \hat{\mathbf{P}}_t'\mathbf{K}_t^+)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^+ \geq -1\}}\right], \end{aligned} \tag{6}$$

with terminal condition $C_T = D_T = 1$. *Moreover, the value function of* $(\mathcal{A}(\mu))$ *at time* t *is given by*

$$J_t(y_t) = D_t\rho_t^2y_t^2\mathbf{1}_{\{y_t\geq 0\}} + C_t\rho_t^2y_t^2\mathbf{1}_{\{y_t<0\}}, \tag{7}$$

where $0 \leq C_t \leq 1$ *and* $0 \leq D_t \leq 1$ *for* $t = 0, \ldots, T-1$.

Proof We mainly use the dynamic programming and mathematical induction method to prove this result. Define the value function of $(\mathcal{A}(\mu))$ at time t as

$$J_t(y_t) \triangleq \min_{\hat{\mathbf{u}}_t\geq\mathbf{0}_{2n},\ldots,\hat{\mathbf{u}}_{T-1}\geq\mathbf{0}_{2n}} \mathrm{E}\left[y_T^2 \,|\, y_t\right]$$

with the terminal condition, $J_T(y_T) = y_T^2$. The statement (7) is true for time T, by setting $C_T = D_T = 1$ in (7). Assume that statement (7) is true for time $t+1$ with $0 \leq C_{t+1} \leq 1$ and $0 \leq D_{t+1} \leq 1$. We now prove that the statement is also true for time t with $0 \leq C_t \leq 1$ and $0 \leq D_t \leq 1$. From the principle of optimality, the value function $J_t(y_t)$ satisfies the recursion,

$$\begin{aligned} J_t(y_t) &= \min_{\hat{\mathbf{u}}_t\geq\mathbf{0}_{2n}} \mathrm{E}\left[C_{t+1}\rho_{t+1}^2y_{t+1}^2\mathbf{1}_{\{y_{t+1}<0\}} + D_{t+1}\rho_{t+1}^2y_{t+1}^2\mathbf{1}_{\{y_{t+1}\geq 0\}}|y_t\right] \\ &= \min_{\hat{\mathbf{u}}_t\geq\mathbf{0}_{2n}} \rho_{t+1}^2 g_t(\hat{\mathbf{u}}_t, y_t), \end{aligned} \tag{8}$$

where

$$\begin{aligned} &g_t(\hat{\mathbf{u}}_t, y_t) \\ &\quad\triangleq \mathrm{E}\left[C_{t+1}(s_ty_t + \hat{\mathbf{P}}_t'\hat{\mathbf{u}}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\hat{\mathbf{u}}_t<-s_ty_t\}} + D_{t+1}(s_ty_t + \hat{\mathbf{P}}_t'\hat{\mathbf{u}}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\hat{\mathbf{u}}_t\geq -s_ty_t\}}\right]. \end{aligned}$$

Note that the expectation is taken with respect to $\hat{\mathbf{P}}_t$. It is not hard to see that the functional $g_t(\hat{\mathbf{u}}_t, y_t)$ is convex with respect to $\hat{\mathbf{u}}_t$.

Now, we consider three cases. (i) When $y_t < 0$, identifying optimal $\hat{\mathbf{u}}_t$ within the convex cone $\hat{\mathbf{u}}_t \geq \mathbf{0}_{2n}$ is equivalent to identifying optimal $\mathbf{K}_t$ within the convex cone $\mathbf{K}_t \geq \mathbf{0}_{2n}$ when we set $\hat{\mathbf{u}}_t = -s_t\mathbf{K}_ty_t$. In general, $\mathbf{K}_t$ is y_t-dependent. We thus have

$$J_t(y_t) = \min_{\hat{\mathbf{u}}_t\geq\mathbf{0}_{2n}} \rho_{t+1}^2 g_t(\hat{\mathbf{u}}_t, y_t) = \min_{\mathbf{K}_t\geq\mathbf{0}_{2n}} \rho_t^2y_t^2h_t^-(\mathbf{K}_t),$$

where

$$h_t^-(\mathbf{K}_t) \triangleq \mathrm{E}\left[C_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t<1\}} + D_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t\geq 1\}}\right]. \tag{9}$$

Noticing the form of $h_t^-(\mathbf{K}_t)$, it is easy to see that the minimizer of $h_t^-(\mathbf{K}_t)$, $\mathbf{K}_t^-$, does not depend on the state variable y_t. Substituting $\hat{\mathbf{u}}_t^* = -s_t y_t \mathbf{K}_t^-$ back to the value function (8) leads to $J_t(y_t) = C_t \rho_t^2 y_t^2$.

We further prove $0 \le C_t \le 1$. It is easy to see that C_t is the expectation of a piecewise quadratic function of $\hat{\mathbf{P}}_t$. Based on its expression, the fact that $0 \le C_{t+1} \le 1$, $0 \le D_{t+1} \le 1$ implies $C_t \ge 0$. Then, we prove $C_t \le 1$ by showing $C_t \le C_{t+1}$. To do this, we need to invoke Lagrangian duality theory to derive $\mathbf{K}_t^-$. Define the Lagrangian function by introducing the multiplier vector $\lambda_t \ge \mathbf{0}_{2n}$, $L_t(\mathbf{K}_t, \lambda_t) = h_t^-(\mathbf{K}_t) - \lambda_t' \mathbf{K}_t$. The optimal $\mathbf{K}_t^-$ can be expressed by

$$\mathbf{K}_t^- = \arg\min_{\mathbf{K}_t \in \mathbb{R}^{2n}} L_t(\mathbf{K}_t, \lambda_t^*), \tag{10}$$

where the optimal multiplier vector λ_t^* is given by

$$\lambda_t^* = \arg\max_{\lambda_t \ge \mathbf{0}_{2n}} \left\{ \min_{\mathbf{K}_t \in \mathbb{R}^{2n}} L_t(\mathbf{K}_t, \lambda_t) \right\}.$$

The optimizer defined in (10), $\mathbf{K}_t^-$, satisfies the first order optimality condition,

$$\frac{dh_t^-(\mathbf{K}_t)}{d\mathbf{K}_t}\Big|_{\mathbf{K}_t = \mathbf{K}_t^-} - \lambda_t^* = \mathbf{0}_{2n}.$$

Define

$$f(\hat{\mathbf{P}}_t, \mathbf{K}_t) \triangleq C_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}} + D_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \ge 1\}}.$$

Consider the open spherical neighbourhood of $\mathbf{K}_t^-$, $\{\mathbf{K}_t \mid ||\mathbf{K}_t - \mathbf{K}_t^-||_2 < \epsilon\}$ for some $\epsilon > 0$, where $||\cdot||_2$ denotes the Euclidean norm of a vector. It is easy to check that in the spherical neighbourhood of $\mathbf{K}_t^-$, $f(\hat{\mathbf{P}}_t, \mathbf{K}_t)$ is a Lebesgue-integrable function of $\hat{\mathbf{P}}_t$ and

$$\begin{aligned}
&\left|\frac{\partial f(\hat{\mathbf{P}}_t, \mathbf{K}_t)}{\partial K_t^i}\right| \\
&= 2\left|C_{t+1}(\hat{P}_t^i \hat{\mathbf{P}}_t'\mathbf{K}_t - \hat{P}_t^i)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}} + D_{t+1}(\hat{P}_t^i \hat{\mathbf{P}}_t'\mathbf{K}_t - \hat{P}_t^i)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \ge 1\}}\right| \\
&\le 2\left|C_{t+1}(\hat{P}_t^i \hat{\mathbf{P}}_t'\mathbf{K}_t - \hat{P}_t^i)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}}\right| + \left|D_{t+1}(\hat{P}_t^i \hat{\mathbf{P}}_t'\mathbf{K}_t - \hat{P}_t^i)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \ge 1\}}\right| \\
&\le 2\left|(\hat{P}_t^i \hat{\mathbf{P}}_t'\mathbf{K}_t - \hat{P}_t^i)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}}\right| + \left|(\hat{P}_t^i \hat{\mathbf{P}}_t'\mathbf{K}_t - \hat{P}_t^i)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \ge 1\}}\right| \\
&= 2\left|\hat{P}_t^i \hat{\mathbf{P}}_t'\mathbf{K}_t - \hat{P}_t^i\right| \\
&\le 2|\hat{P}_t^i| \cdot (|\hat{\mathbf{P}}_t'\mathbf{K}_t| + 1) \\
&\le 2|\hat{P}_t^i| \cdot (||\hat{\mathbf{P}}_t||_2 \cdot (||\mathbf{K}_t^-||_2 + \epsilon) + 1) \\
&\triangleq \theta_i(\hat{\mathbf{P}}_t), \quad i = 1, \ldots, n,
\end{aligned}$$

where $|\cdot|$ denotes the operator of taking absolute value, $\hat{P}_t^i$ and K_t^i are the i-th components of vectors $\hat{\mathbf{P}}_t$ and $\mathbf{K}_t$, respectively. Then, $|\frac{\partial f(\hat{\mathbf{P}}_t,\mathbf{K}_t)}{\partial K_t^i}|$ is bounded by an integrable function $\theta_i(\hat{\mathbf{P}}_t)$. Lebesgue Dominated Convergence Theorem ensures that the differentiation and the expectation are interchangeable at $\mathbf{K}_t = \mathbf{K}_t^-$, i.e.,

$$\begin{aligned}
&\frac{dh_t^-(\mathbf{K}_t)}{d\mathbf{K}_t}\Big|_{\mathbf{K}_t=\mathbf{K}_t^-} - \lambda_t^* \\
&= \mathrm{E}\left[\frac{\partial f(\hat{\mathbf{P}}_t,\mathbf{K}_t)}{\partial \mathbf{K}_t}\Big|_{\mathbf{K}_t=\mathbf{K}_t^-}\right] - \lambda_t^* \\
&= 2\mathrm{E}\Big[C_{t+1}(\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^- - \hat{\mathbf{P}}_t)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}} + D_{t+1}(\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^- - \hat{\mathbf{P}}_t)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}}\Big] - \lambda_t^* \\
&= \mathbf{0}_{2n}.
\end{aligned} \tag{11}$$

Then we further have

$$\begin{aligned}
C_t &= \mathrm{E}\Big[C_{t+1}(1 - 2\hat{\mathbf{P}}_t'\mathbf{K}_t^- + (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}} \\
&\quad + D_{t+1}(1 - 2\hat{\mathbf{P}}_t'\mathbf{K}_t^- + (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}}\Big] \\
&= \mathrm{E}\Big[C_{t+1}(1 - (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}} + D_{t+1}(1 - (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}}\Big] \\
&\quad + 2(\mathbf{K}_t^-)'\mathrm{E}\Big[C_{t+1}(\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^- - \hat{\mathbf{P}}_t)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}} + D_{t+1}(\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^- - \hat{\mathbf{P}}_t)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}}\Big] \\
&= \mathrm{E}\Big[C_{t+1}(1 - (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}} + D_{t+1}(1 - (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}}\Big] \\
&\quad + (\mathbf{K}_t^-)'\lambda_t^* \\
&= \mathrm{E}\Big[C_{t+1}(1 - (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}} + D_{t+1}(1 - (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}}\Big] \\
&\leq C_{t+1}\mathrm{E}\Big[(1 - (\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}\Big] \\
&= C_{t+1}\mathrm{E}\Big[\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}] - C_{t+1}\mathrm{E}[(\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}\Big] \\
&\leq C_{t+1},
\end{aligned}$$

where the first order optimality condition given in (11) is used to derive the third equality, the complementary slackness condition $(\lambda_t^*)'\mathbf{K}_t^- = 0$ is used to drive the fourth equality, $0 \leq D_{t+1} \leq 1$ is used to derive the first inequality, $0 \leq C_{t+1} \leq 1$ and $\mathrm{E}[\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}] \leq 1$ are used to drive the second inequality. We can further see that $C_t = C_{t+1}$ if and only if $\mathbf{K}_t^- = \mathbf{0}_{2n}$.

(ii) We then consider the case of $y_t > 0$. Similarly, identifying optimal $\hat{\mathbf{u}}_t \geq \mathbf{0}_{2n}$ is equivalent to identifying optimal $\mathbf{K}_t$ within the convex cone $\mathbf{K}_t \geq \mathbf{0}_{2n}$ when we set $\hat{\mathbf{u}}_t = s_t\mathbf{K}_t y_t$. In general, $\mathbf{K}_t$ is y_t-dependent. We thus have

$$J_t(y_t) = \min_{\hat{\mathbf{u}}_t \geq \mathbf{0}_{2n}} \rho_{t+1}^2 g_t(\hat{\mathbf{u}}_t, y_t) = \min_{\mathbf{K}_t \geq \mathbf{0}_{2n}} \rho_t^2 y_t^2 h_t^+(\mathbf{K}_t),$$

where

$$h_t^+(\mathbf{K}_t) \triangleq \mathrm{E}[C_{t+1}(1+\hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t<-1\}} + D_{t+1}(1+\hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \geq -1\}}]. \tag{12}$$

Noticing the form of $h_t^+(\mathbf{K}_t)$, it is easy to see that the minimizer of $h_t^+(\mathbf{K}_t)$, $\mathbf{K}_t^+$, does not depend on the state variable y_t. Substituting $\hat{\mathbf{u}}_t^* = s_t y_t \mathbf{K}_t^+$ back to the value function (8) leads to $J_t(y_t) = D_t \rho_t^2 y_t^2$. We can also show that $0 \leq D_t \leq 1$ by using the similar argument for C_t. Furthermore, we have that $D_t = D_{t+1}$ if and only if $\mathbf{K}_t^+ = \mathbf{0}_{2n}$.

(iii) When $y_t = 0$, we can easily verify that $\hat{\mathbf{u}}_t^* = \mathbf{0}_{2n}$ is the minimizer of $g_t(\hat{\mathbf{u}}_t, 0)$, which completes the proof of the Theorem. □

The parameters C_t and D_t are the coefficients of the cost-to-go functions of problem $(\mathcal{A}(\mu))$. Furthermore, by using Theorem 1 and Proposition 1, we achieve the semi-analytical optimal policy of problem $(\mathcal{P}(b))$.

Theorem 2 *When $C_0 = 1$, the optimal portfolio policy of problem $(\mathcal{P}(b))$ is*

$$\mathbf{u}_t^* = \mathbf{0}_n, \quad \mathbf{v}_t^* = \mathbf{0}_n.$$

When $C_0 < 1$, the optimal portfolio policy of problem $(\mathcal{P}(b))$ is expressed by

$$\begin{pmatrix} \mathbf{u}_t^* \\ \mathbf{v}_t^* \end{pmatrix} = s_t \mathbf{K}_t^+ \left(x_t - \rho_t^{-1}(b-\mu^*)\right) \mathbf{1}_{\{x_t \geq \rho_t^{-1}(b-\mu^*)\}} - s_t \mathbf{K}_t^- \left(x_t - \rho_t^{-1}(b-\mu^*)\right) \mathbf{1}_{\{x_t < \rho_t^{-1}(b-\mu^*)\}}, \tag{13}$$

where $\mu^* = \frac{C_0(b-\rho_0 x_0)}{C_0 - 1}$. Moreover, the mean–variance efficient frontier is

$$\mathrm{Var}[x_T] = \frac{C_0\left(\mathrm{E}[x_T] - \rho_0 x_0\right)^2}{1 - C_0}, \quad \text{for } \mathrm{E}[x_T] \geq \rho_0 x_0. \tag{14}$$

Proof From Proposition 1, we know that the optimal objective value of problem $(\mathcal{A}(\mu))$ is

$$v(\mathcal{A}(\mu)) = C_0(\gamma - \rho_0 x_0)^2 \mathbf{1}_{\{\gamma > \rho_0 x_0\}} + D_0(\gamma - \rho_0 x_0)^2 \mathbf{1}_{\{\gamma \leq \rho_0 x_0\}}, \tag{15}$$

where $\gamma = b - \mu$. As $v(\mathcal{A}(\mu)) = v(\mathcal{L}(\mu)) + \mu^2$, we have

$$v(\mathcal{L}(\mu)) = C_0(\gamma - \rho_0 x_0)^2 \mathbf{1}_{\{\gamma > \rho_0 x_0\}} + D_0(\gamma - \rho_0 x_0)^2 \mathbf{1}_{\{\gamma \leq \rho_0 x_0\}} - \mu^2.$$

Theorem 1 implies that optimal μ can be found by solving the following problem,

$$\mu^* = \arg\max_{\mu\in\mathbb{R}} \left\{ \left(C_0 \mathbf{1}_{\{b-\mu>\rho_0 x_0\}} + D_0 \mathbf{1}_{\{b-\mu\le\rho_0 x_0\}} \right)(b-\mu-\rho_0 x_0)^2 - \mu^2 \right\}.$$

Please note that $b \ge x_0 \prod_{t=0}^{T-1} s_t = \rho_0 x_0$.

When $C_0 = 1$, we have $\mu^* = -\infty$ and $\mathbf{K}_t^- = \mathbf{0}_{2n}$. Then, the investor starts from the domain $x_0 < \rho_0^{-1}(b-\mu^*)$ and takes $\mathbf{u}_0^* = \mathbf{0}_n$, $\mathbf{v}_0^* = \mathbf{0}_n$ as the portfolio policy at time 0. When comes to time 1, the investor remains in that domain $x_1 = s_0 x_0 < \rho_1^{-1}(b-\mu^*)$ and also takes $\mathbf{u}_1^* = \mathbf{0}_n$, $\mathbf{v}_1^* = \mathbf{0}_n$ as the portfolio policy at time 1. Applying similar arguments to the terminal time, we can see that the investor takes $\mathbf{u}_t^* = \mathbf{0}_n$, $\mathbf{v}_t^* = \mathbf{0}_n$ as the portfolio policy for all time periods. Thus, $C_0 = 1$ is a degenerate case.

When $C_0 < 1$, it is not hard to identify the optimal μ^* as $\mu^* = \frac{C_0(b-\rho_0 x_0)}{C_0 - 1} \le 0$. Substituting μ^* to the expression of $\mathrm{Var}[x_T]$ gives the efficient frontier (14). □

Next proposition discusses the properties of the semi-analytical optimal policy of problem $(\mathcal{P}(b))$ and reveals the irrational feature of dynamic mean–variance criteria as we mentioned before. Recall that

$$\begin{pmatrix} \mathbf{u}_t^* \\ \mathbf{v}_t^* \end{pmatrix} = (u_t^{1*}, \dots, u_t^{n*}, v_t^{1*}, \dots, v_t^{n*})',$$

where u_t^{i*} and v_t^{i*} are the optimal long and short positions in the i-th risky fund.

Proposition 2 *At time $t = 0, 1, \dots, T-1$, if $x_t < \rho_t^{-1}(b-\mu^*)$, the mean–variance investor would like to take either long or short position in each risky fund, i.e.,*

$$u_t^{i*} v_t^{i*} = 0, \quad i = 1, 2, \dots, n.$$

At time $t = 0, 1, \dots, T-1$, if $x_t \ge \rho_t^{-1}(b-\mu^*)$, the mean–variance investor would like to take the same long and short positions in each risky fund, i.e.,

$$u_t^{i*} = v_t^{i*}, \quad i = 1, 2, \dots, n.$$

Proof Denote

$$\mathbf{K}_t^+ = (k_t^{1+}, \dots, k_t^{n+}, l_t^{1+}, \dots, l_t^{n+})', \quad \mathbf{K}_t^- = (k_t^{1-}, \dots, k_t^{n-}, l_t^{1-}, \dots, l_t^{n-})'.$$

With the help of Theorem 2, the proposition can be proved by showing

$$\begin{aligned} &k_t^{i-} l_t^{i-} = 0, \quad i = 1, 2, \dots, n, \ t = 0, 1, \dots, T-1, \\ &k_t^{i+} = l_t^{i+}, \quad i = 1, 2, \dots, n, \ t = 0, 1, \dots, T-1. \end{aligned}$$

Now, we prove $k_t^{i+} = l_t^{i+}$, $i = 1, 2, \dots, n$, $t = 0, 1, \dots, T-1$. Recall that at time t, the optimal parameter vector $\mathbf{K}_t^+$ is the minimizer of

$$h_t^+(\mathbf{K}_t) = \mathrm{E}\left[C_{t+1}(1+\hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t<-1\}} + D_{t+1}(1+\hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t\ge-1\}} \right].$$

As $h_t^+(\mathbf{K}_t)$ is a sum of expectations of quadratic functions with non-negative coefficients C_{t+1} and D_{t+1}, it suffices to observe that $h_t^+(\mathbf{K}_t) \geq 0$. When $C_{t+1} = D_{t+1} = 0$, any feasible $\mathbf{K}_t(\geq \mathbf{0}_{2n})$ is the minimizer and we can choose a particular $\mathbf{K}_t^+$ with $k_t^{i+} = l_t^{i+}$ $(i = 1, 2, \ldots, n)$. When $(C_{t+1}, D_{t+1}) \neq (0, 0)$, if there exists $\mathbf{K}_t \geq \mathbf{0}_{2n}$ such that $\hat{\mathbf{P}}_t'\mathbf{K}_t = -1$, then $h_t^+(\mathbf{K}_t) = 0$. To construct such $\mathbf{K}_t$, we can impose $k_t^{i+} = l_t^{i+}$. Then we have

$$\hat{\mathbf{P}}_t'\mathbf{K}_t = \sum_{i=1}^{n}(-s_t)k_t^{i+}(c_t^i + d_t^i),$$

and one can easily find $k_t^{i+} \geq 0, i = 1, \ldots, n$ satisfying $\hat{\mathbf{P}}_t'\mathbf{K}_t = -1$.

To prove $k_t^{i-}l_t^{i-} = 0$, $i = 1, 2, \ldots, n$, $t = 0, 1, \ldots, T-1$, we need the fact that $C_t > 0$, $D_t = 0$, $t = 0, 1, \ldots, T-1$. Firstly, based on the above discussion on $\mathbf{K}_t^+$, we have $h_t^+(\mathbf{K}_t^+) = 0$, which implies

$$D_t = \min_{\mathbf{K}_t \geq \mathbf{0}_{2n}} h_t^+(\mathbf{K}_t) = h_t^+(\mathbf{K}_t^+) = 0, \quad t = 0, 1, \ldots, T-1.$$

Secondly, we prove $C_t > 0$, $t = 0, 1, \ldots, T-1$ by backward induction. At time T, $C_T = 1 > 0$. Now we assume $C_{t+1} > 0$. Consider the following inequality,

$$\hat{\mathbf{P}}_t'(\omega)\mathbf{K}_t = \sum_{i=1}^{n}[P_t^i(\omega)(k_t^i - l_t^i) - s_t(c_t^i k_t^i + d_t^i l_t^i)] \geq 1, \quad \forall\, \omega \in \Omega. \tag{16}$$

We show that the inequality (16) does not admit solution in the convex cone $\mathbf{K}_t \geq \mathbf{0}_{2n}$ by contradiction. Suppose that $\mathbf{K}_t^*$ $(\geq \mathbf{0}_{2n})$ satisfies the inequality (16). Then, the investor can adopt the portfolio policy $\begin{pmatrix}\mathbf{u}_t \\ \mathbf{v}_t\end{pmatrix} = \mathbf{K}_t^*$ at time t to gain almost surely positive wealth level (at least 1) with zero initial wealth, i.e., $x_{t+1}(\omega) = s_t \cdot 0 + \hat{\mathbf{P}}_t'(\omega)\mathbf{K}_t^* \geq 1$, $\forall\, \omega \in \Omega$. This is obviously an arbitrage opportunity. Thus, the inequality (16) should NOT admit solution in the convex cone $\mathbf{K}_t \geq \mathbf{0}_{2n}$, which further implies that the probability of event $\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}$ is positive, i.e.,

$$Pr(\hat{\mathbf{P}}_t'(\omega)\mathbf{K}_t < 1) > 0.$$

Therefore, noticing that $D_{t+1} = 0$, $C_{t+1} > 0$ for $t = 0, 1, \ldots, T-2$, $D_T = C_T = 1$ and $Pr(\hat{\mathbf{P}}_t'(\omega)\mathbf{K}_t < 1) > 0$ for $t = 0, 1, \ldots, T-1$, we have

$$\begin{aligned}
C_t &= \min_{\mathbf{K}_t \geq \mathbf{0}_{2n}} \mathrm{E}\Big[C_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}} + D_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t \geq 1\}}\Big] \\
&\geq \min_{\mathbf{K}_t \geq \mathbf{0}_{2n}} \mathrm{E}\Big[C_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2 \mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t < 1\}}\Big] \\
&> 0, \quad t = 0, 1, \ldots, T-1.
\end{aligned}$$

Furthermore, to prove $k_t^{i-}l_t^{i-}=0,\ i=1,2,\dots,n,\ t=0,1,\dots,T-1$, we also need the following equation, for $t=0,1,\dots,T-1$,

$$\mathrm{E}\left[C_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}+D_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}}\right]=C_t. \tag{17}$$

According to the proof of Proposition 1, we have the following two equations,

$$(\mathbf{K}_t^-)'\mathrm{E}\left[(C_{t+1}\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}+D_{t+1}\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}})\cdot(\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^--\hat{\mathbf{P}}_t)\right]=0,$$
$$\mathrm{E}\left[(C_{t+1}\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}+D_{t+1}\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-\geq 1\}})\cdot(1-(\mathbf{K}_t^-)'\hat{\mathbf{P}}_t\hat{\mathbf{P}}_t'\mathbf{K}_t^-)\right]=C_t,$$

which implies Eq. (17).

Finally, we prove $k_t^{i-}l_t^{i-}=0,\ i=1,2,\dots,n,\ t=0,1,\dots,T-1$ by contradiction. Suppose that $k_t^{i-}l_t^{i-}\neq 0$. We can choose a positive number $z_t=\min\{k_t^{i-},l_t^{i-},2C_ts_t^{-1}(c_t^i+d_t^i)^{-1}\}$, and define

$$\hat{\mathbf{K}}_t=(k_t^{1-},\dots,\hat{k}_t^i,\dots,k_t^{n-},l_t^{1-},\dots,\hat{l}_t^i,\dots,l_t^{n-})'\geq \mathbf{0}_{2n}$$

with $\hat{k}_t^i=k_t^{i-}-z_t$, $\hat{l}_t^i=l_t^{i-}-z_t$. It is easy to compute that

$$\begin{aligned}
h_t^-(\hat{\mathbf{K}}_t)=&\,\mathrm{E}\left[C_{t+1}(1-\hat{\mathbf{P}}_t'\hat{\mathbf{K}}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\hat{\mathbf{K}}_t<1\}}+D_{t+1}(1-\hat{\mathbf{P}}_t'\hat{\mathbf{K}}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\hat{\mathbf{K}}_t\geq 1\}}\right]\\
=&\,\mathrm{E}\Big[C_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^--s_t(c_t^i+d_t^i)z_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-+s_t(c_t^i+d_t^i)z_t<1\}}\\
&\,+D_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^--s_t(c_t^i+d_t^i)z_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-+s_t(c_t^i+d_t^i)z_t\geq 1\}}\Big].
\end{aligned}$$

For time $t=0,1,\dots,T-2$, we have $D_{t+1}=0$, which implies

$$\begin{aligned}
h_t^-(\hat{\mathbf{K}}_t)=&\ \mathrm{E}\left[C_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^--s_t(c_t^i+d_t^i)z_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-+s_t(c_t^i+d_t^i)z_t<1\}}\right]\\
<&\ \mathrm{E}\left[C_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^--s_t(c_t^i+d_t^i)z_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}\right]\\
=&\ \mathrm{E}\left[C_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^-)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}\right]\\
&\ -2\mathrm{E}\left[C_{t+1}(1-\hat{\mathbf{P}}_t'\mathbf{K}_t^-)s_t(c_t^i+d_t^i)z_t\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}\right]\\
&\ +\mathrm{E}\left[C_{t+1}s_t^2(c_t^i+d_t^i)^2z_t^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}\right]\\
=&\ h_t^-(\mathbf{K}_t^-)-2C_ts_t(c_t^i+d_t^i)z_t+C_{t+1}\mathrm{E}\left[\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^-<1\}}\right]s_t^2(c_t^i+d_t^i)^2z_t^2\\
<&\ h_t^-(\mathbf{K}_t^-),
\end{aligned}$$

where $\{\hat{\mathbf{P}}_t'\mathbf{K}_t^- + s_t(c_t^i + d_t^i)z_t < 1\} \subset \{\hat{\mathbf{P}}_t'\mathbf{K}_t^- < 1\}$ is used to derive the first inequality, the equality (17) is used to derive the third equality, $0 < C_{t+1} \leq 1$, $\mathrm{E}[\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t^- < 1\}}] < 1$ and $z_t \leq 2C_t s_t^{-1}(c_t^i + d_t^i)^{-1}$ are used to derive the last inequality. For time $T-1$, we have

$$\begin{aligned} & h_{T-1}^-(\hat{\mathbf{K}}_{T-1}) \\ & \quad = \mathrm{E}[(1 - \hat{\mathbf{P}}_{T-1}'\mathbf{K}_{T-1}^- - s_{T-1}(c_{T-1}^i + d_{T-1}^i)z_{T-1})^2] \\ & \quad = \mathrm{E}[(1 - \hat{\mathbf{P}}_{T-1}'\mathbf{K}_{T-1}^-)^2] - 2\mathrm{E}[(1 - \hat{\mathbf{P}}_{T-1}'\mathbf{K}_{T-1}^-)s_{T-1}(c_{T-1}^i + d_{T-1}^i)z_{T-1}] \\ & \quad + s_{T-1}^2(c_{T-1}^i + d_{T-1}^i)^2 z_{T-1}^2 \\ & \quad = h_{T-1}^-(\mathbf{K}_{T-1}^-) - 2C_{T-1}s_{T-1}(c_{T-1}^i + d_{T-1}^i)z_{T-1} + s_{T-1}^2(c_{T-1}^i + d_{T-1}^i)^2 z_{T-1}^2 \\ & \quad < h_{T-1}^-(\mathbf{K}_{T-1}^-), \end{aligned}$$

where the equality (17) is used to derive the third equality and $z_{T-1} \leq 2C_{T-1}s_{T-1}^{-1}(c_{T-1}^i + d_{T-1}^i)^{-1}$ is used to derive the last inequality. Obviously, the derived inequality $h_t^-(\hat{\mathbf{K}}_t) < h_t^-(\mathbf{K}_t^-)$ contradicts the assumption that $\mathbf{K}_t^-$ is minimizer of $h_t^-(\mathbf{K}_t)$. Thus, we have $k_t^{i-} l_t^{i-} = 0$. □

Proposition 2 has revealed the irrational feature of dynamic mean–variance criteria. When the current wealth level is bigger than the threshold $\rho_t^{-1}(b - \mu^*)$, the mean–variance investor becomes irrational and tries to take both the same long position and short position in a single risky fund in order to reduce his total wealth. The reason behind the irrational behaviors is the non-monotonicity of the quadratic utility. Theorem 1 has shown that the mean–variance criteria is equivalent to a particular quadratic utility maximization problem, and the axis of symmetry of the time t value function is just at the threshold. When the wealth level is bigger than the threshold, the more wealth level, the less utility. Thus, the best action is to reduce the wealth level x_t to the threshold level $\rho_t^{-1}(b - \mu^*)$ with certainty.

Before closing this section, we discuss the computation issue of the optimal parameter vectors $\mathbf{K}_t^\pm$. As revealed in Theorem 2 and Proposition 1, finding the optimal parameter vectors $\mathbf{K}_t^\pm$ plays a key role in identifying the optimal policy $\mathbf{u}_t^*, \mathbf{v}_t^*$. For $h_t^\pm(\mathbf{K}_t)$ defined in (9) and (12) are convex functions with respect to $\mathbf{K}_t$, the optimization problems in (5) and (6) are standard convex optimization problems, which try to minimize convex functions over a convex cone set. The theoretical analysis on these problems can be found in Section 27 of Rockafellar (1970) and the numerical methods suitable for these problems can be found in Chapter 2 of Bertsekas (1999). In the example of next section, we use the sequential quadratic programming algorithm embedded in Matlab function `fmincon` to derive the vectors $\mathbf{K}_t^\pm$.

4 Illustrative example

In this section, we use two illustrative examples to show the procedure of solving problem ($\mathcal{P}(b)$), analyze how the management fee affects the investment performance, and confirm the theoretical finding in Proposition 2.

Example 1 In this example, there are $n = 10$ funds, which are the first 10 industry indices out of the 48 industry portfolios constructed by Fama and French.[4] The expected monthly return vector and the covariance matrix of these 10 funds' return are estimated by using the historical monthly return from Jan 1998 to Dec 2015,[5] which are given as follows,

$$
\begin{aligned}
&\mathrm{E}[\mathbf{e}_t] = (1.0072, 1.0052, 1.0074, 1.0054, 1.0096, 1.0026, 1.0094, 1.0030, 1.0046, 1.0099)',\\
&\mathrm{Cov}[\mathbf{e}_t]\\
&= \begin{pmatrix}
0.0047, 0.0007, 0.0008, 0.0007, 0.0008, 0.0014, 0.0021, 0.0016, 0.0008, 0.0016\\
0.0007, 0.0015, 0.0012, 0.0010, 0.0012, 0.0011, 0.0014, 0.0010, 0.0009, 0.0012\\
0.0008, 0.0012, 0.0055, 0.0017, 0.0013, 0.0019, 0.0026, 0.0019, 0.0014, 0.0021\\
0.0007, 0.0010, 0.0017, 0.0022, 0.0010, 0.0011, 0.0013, 0.0009, 0.0011, 0.0011\\
0.0008, 0.0012, 0.0013, 0.0010, 0.0051, 0.0014, 0.0015, 0.0010, 0.0009, 0.0010\\
0.0014, 0.0011, 0.0019, 0.0011, 0.0014, 0.0043, 0.0034, 0.0022, 0.0014, 0.0028\\
0.0021, 0.0014, 0.0026, 0.0013, 0.0015, 0.0034, 0.0069, 0.0035, 0.0017, 0.0037\\
0.0016, 0.0010, 0.0019, 0.0009, 0.0010, 0.0022, 0.0035, 0.0037, 0.0013, 0.0026\\
0.0008, 0.0009, 0.0014, 0.0011, 0.0009, 0.0014, 0.0017, 0.0013, 0.0018, 0.0013\\
0.0016, 0.0012, 0.0021, 0.0011, 0.0010, 0.0028, 0.0037, 0.0026, 0.0013, 0.0042
\end{pmatrix}.
\end{aligned}
$$

The investment horizon is set as $T = 3$ months and the management fee rate of each fund is set as $c_t^i = d_t^i = \xi$ for $t = 0, 1, 2$, $i = 1, 2, \dots, 10$, where the parameter ξ is chosen from set $\{0, 0.001, 0.002, 0.003, 0.004\}$. The monthly return of the riskless bank account is set as $s_t = 1.001$ for $t = 0, 1, 2$.

The investor with initial wealth $x_0 = 1$ adopts the mean–variance model $(\mathcal{P}(b))$ to decide his optimal portfolio policy. To solve the optimization problems (5) and (6), we adopt the Monte Carlo method to approximate the expected value in objective function. More specifically, in any time period, we assume the return vector $\mathbf{e}_t$ follows multivariate normal distribution and generate $N = 50{,}000$ samples of $\mathbf{e}_t$ according to $\mathrm{E}[\mathbf{e}_t]$ and $\mathrm{Cov}[\mathbf{e}_t]$. Then, we can get N samples of $\mathbf{P}_t$ according to its definition, which are denoted as $\mathbf{P}_t(i)$, $i = 1, 2, \dots, N$. Taking the objective function of (5) as an example, we can approximate it as,

$$
\begin{aligned}
&\mathrm{E}\left[C_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t<1\}} + D_{t+1}(1 - \hat{\mathbf{P}}_t'\mathbf{K}_t)^2\mathbf{1}_{\{\hat{\mathbf{P}}_t'\mathbf{K}_t\ge 1\}}\right]\\
&\approx \frac{1}{N}\sum_{i=1}^{N}\Big(C_{t+1}(1 - (\mathbf{P}_t(i))'\mathbf{K}_t)^2\mathbf{1}_{\{(\mathbf{P}_t(i))'\mathbf{K}_t<1\}}\\
&\quad + D_{t+1}(1 - (\mathbf{P}_t(i))'\mathbf{K}_t)^2\mathbf{1}_{\{(\mathbf{P}_t(i))'\mathbf{K}_t\ge 1\}}\Big).
\end{aligned}
$$

Then, we call the function `fmincon` in MATLAB to solve the problem, where the sequential quadratic programming algorithm is adopted.

[4] The data of 48 industry portfolios can be found in http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

[5] To achieve a stable estimation, the shrinkage estimation method in Ledoit and Wolf (2003) is used in estimating the covariance matrix.

For different level of management fee rate, $\xi \in \{0, 0.001, \ldots, 0.004\}$, we derive the optimal parameter vectors $\mathbf{K}_t^{\pm}$ and the solution parameters C_t and D_t, $t = 0, 1, 2$, which are used to computed the mean–variance efficient frontier. The optimal parameter vectors $\mathbf{K}_t^{\pm}$ and C_t, D_t at time $t = 2$ for $\xi = 0.001, 0.002, 0.003, 0.004$ are reported as follows,

$$
\begin{aligned}
& C_2|_{\{\xi=0.001\}} = 0.9645,\ \ C_2|_{\{\xi=0.002\}} = 0.9782,\\
& C_2|_{\{\xi=0.003\}} = 0.9858,\ \ C_2|_{\{\xi=0.004\}} = 0.9903,\\
& D_2|_{\{\xi=0.001\}} = D_2|_{\{\xi=0.002\}} = D_2|_{\{\xi=0.003\}} = D_2|_{\{\xi=0.004\}} = 0.0000,\\
& \mathbf{K}_2^{+}|_{\{\xi=0.001\}}\\
& \quad = (47.005, 46.050, 45.914, 46.143, 51.338, 58.554, 46.293, 49.263, 45.861, 63.081,\\
& \qquad 47.005, 46.050, 45.914, 46.143, 51.338, 58.554, 46.293, 49.263, 45.861, 63.081)',\\
& \mathbf{K}_2^{+}|_{\{\xi=0.002\}}\\
& \quad = (24.685, 24.506, 24.518, 24.526, 25.479, 26.692, 24.582, 24.888, 24.465, 25.415,\\
& \qquad 24.685, 24.506, 24.518, 24.526, 25.479, 26.692, 24.582, 24.888, 24.465, 25.415)',\\
& \mathbf{K}_2^{+}|_{\{\xi=0.003\}}\\
& \quad = (16.600, 16.544, 16.565, 16.550, 16.781, 16.965, 16.599, 16.613, 16.531, 16.752,\\
& \qquad 16.600, 16.544, 16.565, 16.550, 16.781, 16.965, 16.599, 16.613, 16.531, 16.752)',\\
& \mathbf{K}_2^{+}|_{\{\xi=0.004\}}\\
& \quad = (12.479, 12.452, 12.468, 12.455, 12.547, 12.534, 12.489, 12.463, 12.446, 12.543,\\
& \qquad 12.479, 12.452, 12.468, 12.455, 12.547, 12.534, 12.489, 12.463, 12.446, 12.543)',\\
& \mathbf{K}_2^{-}|_{\{\xi=0.001\}}\\
& \quad = (0.735, 0.005, 0.274, 0.431, 1.284, 0.000, 0.572, 0.000, 0.293, 2.479,\\
& \qquad 0.000, 0.000, 0.000, 0.000, 0.000, 1.888, 0.000, 1.184, 0.000, 0.000)',\\
& \mathbf{K}_2^{-}|_{\{\xi=0.002\}}\\
& \quad = (0.460, 0.000, 0.156, 0.165, 1.126, 0.000, 0.124, 0.000, 0.000, 1.864,\\
& \qquad 0.000, 0.000, 0.000, 0.000, 0.000, 1.099, 0.000, 0.129, 0.000, 0.000)',\\
& \mathbf{K}_2^{-}|_{\{\xi=0.003\}}\\
& \quad = (0.254, 0.000, 0.032, 0.001, 0.930, 0.000, 0.001, 0.000, 0.000, 1.372,\\
& \qquad 0.000, 0.000, 0.000, 0.000, 0.000, 0.303, 0.000, 0.000, 0.000, 0.000)',\\
& \mathbf{K}_2^{-}|_{\{\xi=0.004\}}\\
& \quad = (0.084, 0.000, 0.000, 0.000, 0.754, 0.000, 0.000, 0.000, 0.000, 1.064,\\
& \qquad 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)'.
\end{aligned}
$$

These results confirm the findings of Proposition 2. Moreover, we can see that when the management fee rate increases, the optimal parameter vectors $\mathbf{K}_t^{\pm}$ decrease and the parameter C_t increases.

In Fig. 1, we present the influence of difference levels of management fees on the mean–variance efficient frontier of the terminal wealth. As we can see from Fig. 1, for a given level of variance, the expected terminal wealth decreases when the management fee rate increases. Moreover, the Sharpe ratio of the efficient frontier

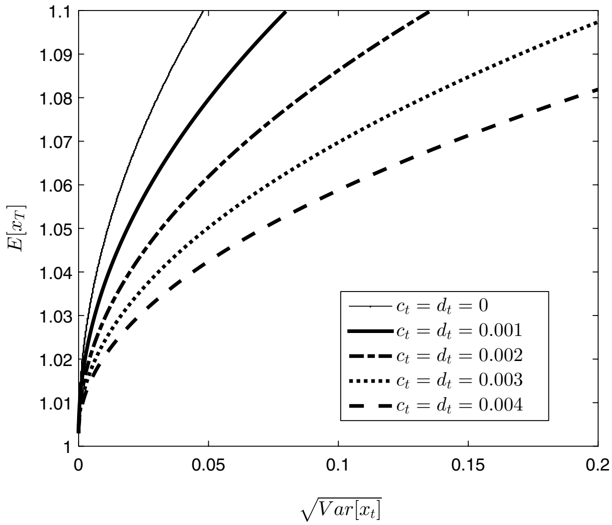

Fig. 1 The efficient frontiers for different levels of management fees

nonlinearly decreases with respect to the management fee rate, as revealed by the solid line in Fig. 2.

Now we focus on analyzing the benefit and cost of conducting dynamic portfolio policy with management fees. The benchmark is that the investor directly invests in the stocks instead of funds and adopts buy-and-hold policy. More specifically, the investor regards the three-month investment model as a single period MV portfolio model and invests the risky funds without management fees by duplicating the funds' portfolios. It is not hard to compute the Sharpe ratio by adopting such a buy-and-hold static MV portfolio policy as 0.4143 (see Markowitz 1952). When the investor adopts a dynamic portfolio policy with management fees for three months, i.e., the investor considers the portfolio model ($\mathcal{P}(b)$). According to Theorem 2, the optimal Sharpe ratio of the terminal wealth is given as

$$\frac{\mathrm{E}[x_3] - \rho_0 x_0}{\sqrt{\mathrm{Var}[x_3]}} = \sqrt{\frac{1 - C_0}{C_0}}.$$

In Fig. 2, the dash line represents the benchmark of the Sharpe ratio achieved by the buy-and-hold static MV portfolio policy, in which no management fees are charged and the solid line represents the Sharpe ratio achieved by the dynamic MV portfolio policy (13) with different levels of management fee rate. When the Sharpe ratio achieved by the dynamic policy is higher than the benchmark (the dash line), it is worth for the investor to adopt dynamic portfolio policy with management fees. Otherwise, it is better for the investor to adopt the buy-and-hold policy himself. For this particular numerical example, when the management fee rate is less than 0.048%

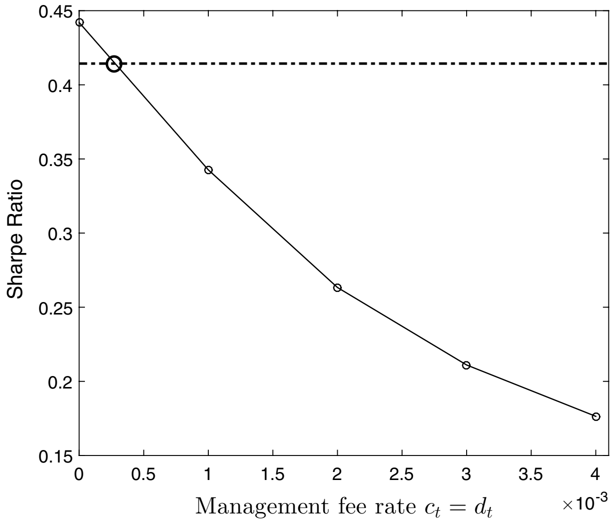

Fig. 2 The Sharpe ratio and management fee rate

(fee rate per month), the investor would be better off by adopting dynamic portfolio policy with management fees.

5 Conclusion

Motivated by the fact that some individual investors prefer to invest in funds instead of stocks, this work studies the multi-period MV portfolio optimization problem with proportional management fees. Under general market setting, we derived the semi-analytical optimal portfolio policy for this problem. Meanwhile the numerical approach is given to compute such optimal portfolio policy. Different from the traditional multi-period MV portfolio policy (see Li and Ng 2000), the revealed portfolio policy is a piecewise affine function with respect to the current wealth. Our model has certain potential to be calibrated by the real market data and used in the empirical tests in the future research related to management fees.

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